New analytical solutions for a flat rounded punch compared with FEM

J. Jäger,
Lauterbach Verfahrenstechnik, Germany

Abstract

Recently, a generalized Coulomb law for elastic bodies in contact was developed by the author and several applications for elastic half-planes, half-spaces, thin and thick layers and impact problems have been published. This theory assumes that the tangential traction is the difference of the slip stress of the contact and the stick area, and each stick area is a smaller contact area. It holds for multiple contact regions also. For plane contact of equal bodies with friction, it provides exact solutions and the interior stress field can be expressed with analytical results in closed form. In this article, a symmetric superposition of flat punch solutions is outlined and some useful formulas are listed. The necessary assumptions and simplifications are discussed. It is shown that this superposition satisfies Coulomb’s inequalities directly and can also be used for torsion or shift of axisymmetric profiles. A new formula for the Muskhelishvili potential of a two-dimensional flat rounded punch is given. Optional load histories can be written as a superposition of unidirectional displacement increments. This generalized Coulomb model is exact for equal half-planes, axisymmetric half-spaces and thin layers [4, 5, 8]. The elastic friction model was also adopted by other researchers, and a discussion was given in [9, 11].

In this article, the elastic Coulomb law is applied to flat rounded punch in contact with an elastic plane and compared with FEM results. A FEM analysis of the Hertz contact problem with friction was published in [1]. Recently, several publications appeared on simplified contact laws. A torque-displacement relation for elliptical contact in torsion by Cuttino and Dow was discussed in [6]. The application of simplified force-displacement relations for granular flow simulations was suggested in [14]. More publications can be found in a review on FEM approaches [12].

The surface displacements $u_x$ and $u_z$ of equal half-planes under the surface loading $q$, $p$ in $x$- and $z$-direction, respectively, can be written as integral of point force displacements [5]

$$
\int \pi \frac{\partial u_x}{\partial x} = -\int a(x) \frac{\partial \xi}{\partial x}, \quad \int \pi \frac{\partial u_z}{\partial x} = -\int p(x) \frac{\partial \xi}{\partial x},
$$

where $A = \frac{\kappa_1 + 1}{4G_1} + \frac{\kappa_2 + 1}{4G_2}, \quad \kappa_1 = \frac{3 - 4v_1}{3 - 4v_1 + 2v_1}$. The symbols $G_1$ and $v_1$ denote the shear modulus and Poisson’s ratio of body 1, respectively, and the index 2 characterizes values of body 2. It was explained in [5] that equations (1) are also exact for a rigid punch in contact with an incompressible elastic half-plane (plane strain, $G_2 = \infty$, $v_1 = 0.5$).

2 Symmetric superposition of flat punch solutions

For the problem of a flat punch with sharp corners in contact with an elastic plane, the solution of the singular integral equations (1) is well known. The pressure $p(a,x)$, the normal displacement $u_x(a,x)$, and the complex Muskhelishvili potential $\Phi_p(a,w)$ have the form [5]

$$
p(a,x) = \frac{P_0}{\pi \sqrt{a^2 - x^2}}, \quad \text{for } x \leq a,
$$

$$
u_x(a,x) = \begin{cases} u_x(a,0), \quad \text{for } x \leq a \\ u_x(a,0) - \frac{P_0}{\pi} \ln \left( \frac{|a|}{x} \sqrt{x^2 - a^2} + a \right), \quad \text{for } |x| \geq a \end{cases}
$$
The contact condition \( p(a,x) > 0 \) for the pressure in the contact area holds for the increments \( dp(a,x) \) also, and from equation (6) follows \( p_0(s) > 0 \) as a necessary condition for contact. Equation (9) on the other hand shows that for the case \( 0 < p_0(s) < \infty \) the slope must be positive: \( \infty > z_1'(x) > 0 \) and vice versa. This means that for profiles with a finite slope \( z_1'(x) > 0 \) the inequality \( p(a,x) > 0 \) is always satisfied. For \( z_1'(x) = 0 \), a rigid body motion in form of a flat punch solution can be superposed. Such a general proof cannot be given for the condition of separation (7), which must be proved separately for each profile.

### 3 Tangential solution

The integral equations in tangential direction (1) are the same as in normal direction and the normal solution can be used for the tangential problem. For simplicity, values of the stick area \( x < a^* < a \) will be denoted with asterisk *, e.g. \( p^* = p(a^*,x) \), in the equations below. It can easily be shown that the pressure difference \( p-p^* \) satisfies the stick condition, which is a constant shift of the stick area [5]. Thus we obtain the tangential traction \( q \), the force \( Q \) and the tangential displacement \( u_t \) with Coulomb’s coefficient of friction \( f \). Coulomb’s inequality in the stick area

\[
q = f(p - p^*), \quad Q = f(P - P^*), \quad u_t = f\kappa(u_0 - u_0^*),
\]

(11)

with Coulomb’s coefficient of friction \( f \). Coulomb’s inequality in the stick area

\[
q = f(p - p^*) < fp, \quad \text{in the stick area},
\]

(12)

is always satisfied, because the contact condition for the normal solution requires positive pressure \( p^* > 0 \) and \( p > 0 \).

The second inequality of Coulomb’s law requires that the slip velocity \( dsdt \) must be opposite to the tangential traction \( q = fp - p^* \). For constant normal pressure \( p \) and decreasing stick areas \( da^* < 0 \), the tangential traction increment \( dq = -fp > 0 \) depends on the stick radius \( a^* \), at a constant position \( x \). The pressure increment \( dp_0 = ds \) is a flat punch solution. The corresponding slip increment \( ds_0 = dp_0 = da^* = du_0(a^*,0) \) of the stick area

\[
d_s(a^*,x) = du_0(a^*,x) - du_0(a^*,0) = f\kappa[du_0(a^*,x) - du_0(a^*,0)],
\]

(13)

where equation (11) was used. Equation (4) shows that the slip \( ds_0(a^*,x) \) must be negative, and opposite to the traction \( q \), as required by Coulomb’s slip inequality. In the case of general load histories, this proof holds also for a superposition of displacement increments. When finite increments are used, as usual in numerical mechanics, it can be shown that Coulomb’s inequality is identical with the condition of separation for the normal problem. The differential formulation (13) is more general, and can be used in the same form for torsion of axisymmetric surfaces [4]. Thus, the validity of Coulomb’s inequalities is a direct consequence of the method of superposition of flat rigid punches.
4 Flat rounded punch

In a series of publications [2], [3], a Chebyshev expansion has been used for the Muskhelishvili potential of a flat rounded punch, and a wedge with rounded tip. This potential is useful for the interior stress field, and a simple analytical solution in closed form is derived below. A comprehensive discussion of [2], [3] was given in [9, 11]. For the special case of a flat punch with rounded corners (Fig. 1), the gap \( z_f(r) \) between the surfaces in undeformed contact has the form

\[
z_f(x) = H \left( \frac{x}{b} \right) \frac{(x-b)^2}{2R_c}, \quad \text{for} \ x \leq a,
\]

\( H(x) = \begin{cases} 0, & x < 1 \\ 1, & x \geq 1 \end{cases} \) (14)

Insertion of (14) in (10) gives

\[
AR_c p_0(s) = H \left( \frac{s}{b} \right) 2 s \arccos \frac{b}{s},
\]

The pressure \( p(a,x) \) in (6) can be evaluated with (15)

\[
\pi AR_c p(a,x) = 2 \sqrt{a^2-x^2} \arccos \frac{b}{a} - b \ln \frac{\sqrt{a^2-b^2} + \sqrt{a^2-x^2}}{|x^2-b^2|}
\]

\[
+ x \ln \left( \frac{b \sqrt{a^2-x^2} + x \sqrt{a^2-b^2}}{a \sqrt{a^2-b^2}} \right). \tag{16}
\]

Numerical comparison shows that equation (16) is identical with Schubert’s formula (20)-(21) in [13]. The normal force is the integral of differential forces \( p_0(s) \)

\[
P(x) = \int_{s=0}^{x} p_0(s) ds = \frac{1}{3AR_c} \left( 2(4a^2-b^2) \sqrt{a^2-b^2} - 6ba \arccos \frac{b}{a} \right)
\]

with the coordinates \( x \) and \( z \). The Muskhelishvili potential \( \phi_p \) can be calculated by insertion of eq. (15) in (6)

\[
\pi AR_c \phi_p(a,w) = -i \sqrt{w^2-a^2} \arccos \frac{b}{a} + i \arcsin \left( \frac{w a^2-b^2}{a \sqrt{w^2-b^2}} \right) - i b \arcsin \left( \frac{a^2-b^2}{\sqrt{w^2-b^2}} \right), \tag{18}
\]

In Muskhelishvili’s definition the z-axis points outside of the body. The derivative \( \partial \phi_p/\partial w \) is necessary for the stress calculation

\[
\pi AR_c \frac{\partial \phi_p}{\partial w} = -iw \arccos \frac{b}{a} + i \arcsin \left( \frac{w a^2-b^2}{a \sqrt{w^2-b^2}} \right). \tag{19}
\]

The displacement \( u_r(a,x) \) follows from equation (4), similar as equation (8)

\[
u_r(a,x) = u_r(a,0) - \frac{A}{\pi} \int_{s=0}^{x} p_0(s) \ln \left( \frac{|s|}{s} + \sqrt{s^2 - 1} \right) ds, \quad F = \text{Min}(x,a). \tag{20}
\]

Equation (20) can be integrated numerically. The Muskhelishvili potential for the tangential loading alone was derived in [5]

\[
\phi_r(a,a*,w) = i \left( \phi_p(a,w) - \phi_p(a*,w) \right), \tag{21}
\]

The total potential for normal and tangential loading is

\[
\phi_{total}(a,a*,w) = (1 + i) \phi_p(a,w) - i \phi_p(a*,w). \tag{22}
\]

Finally, it may be mentioned that the profile \( z_2(x) \) of a wedge with rounded tip can be written as the difference of a Hertzian profile and a flat rounded punch \( z_f \) given by eq. (14)

\[
z_2(x) = \frac{x^2}{2R_c} - z_f(x). \tag{23}
\]

The profile \( z_2(x) \) of the rounded wedge is parabolical for \( b < x \) and has a constant slope for \( b < x < a \). In linear elasticity, solutions can be superposed linearly, and it is not necessary to perform any calculation for the solution of this case. The Hertzian result is the special case \( b=0 \) in equations (14)-(22). As example, the pressure of the wedge with rounded tip is the difference of equation (16) with the same term with \( b=0 \) and the same term with \( b>0 \).

5 Interior stress field

The von Mises stress \( \sigma_z \) can be expressed in terms of the strain energy due to distortion [5]

\[
\sigma_z^2 = \sigma_x^2 + \sigma_y^2 + \sigma_z^2 - (\sigma_x \sigma_y + \sigma_x \sigma_z + \sigma_y \sigma_z) + 3(\tau_x^2 + \tau_y^2 + \tau_z^2).
\]

Plane stress: \( \sigma_y = \tau_y = \tau_z = 0 \)

Plane strain: \( \sigma_y = \nu(\sigma_x + \sigma_z), \quad \tau_y = \tau_z = 0 \)

The stress components can be written as a function of the Muskhelishvili potential

\[
\gamma_1 = \frac{1}{2}(\sigma_x + \sigma_y) = \phi(w) + \phi(\bar{w})
\]

\[
\delta_1 = \frac{1}{2}(\sigma_y - \sigma_x + 2\tau_y) = (\bar{w} - w)\phi'(w) + \phi(\bar{w}) - \phi(w)
\]

The bar denotes conjugate complex values. Insertion of (25) in (24) gives

\[
\sigma_z^2 = \begin{cases} \gamma_1^2 + 3\delta_1 \delta_2, & \text{for plane stress} \\ (1-2\nu^2)\gamma_1^2 + 3\delta_1 \delta_2, & \text{for plane strain} \end{cases} \tag{26}
\]

The complex functions can easily be evaluated with mathematical software packages, and an example for Mathematica was presented in [5].
6 Comparison with FEM

Fig. 2 illustrates the FEM-model, which was solved with ANSYS 5.5. Rigid elements TARGE169 are used to represent the rigid punch with the semi-axis $b=5$ mm of the flat region and the rounding radius $R_c=80$ mm. A circular region of the half-plane is modeled with 2-D elements PLANE42 and is fixed radially at $R_G=50$ mm. The x-axis of the half-plane represents the contact surface and consists of CONTA171 elements, which are associated with the target elements. The half-plane has a modulus of Elasticity $E=1000$ N/mm$^2$, Poisson’s ratio $\nu=0.4999\approx0.5$ and a coefficient of friction $f=0.5$. The normal force is applied in steps 1-3, calculated from equation (17) for the values $a=\{5.4, 5.8, 6.2\}$ of the contact area. Four increments are used for each step. After normal loading, a tangential force is applied in steps 4-6, with the values $a^*=\{5.8, 5.4, 0\}$ for the semi-axis of the stick area. With these values, the normal solution can directly be inserted in eqs. (11) for the tangential problem, e.g. as the traction: $q = f (p(a,x)−p(a^*,x))$.

A comparison between the normal pressure of ANSYS (markers) and the analytical result (full line) is shown in Fig. 4. The difference between both methods is very small. The vertical slope of the pressure $\partial p(a,x)/\partial x = \infty$ at the positions $x=a$ and $x=b$ is very typical for a discontinuous variation of the curvature. The difference between the theoretical and numerical tractions of steps 4-6 in Fig. 5 is larger. Small oscillations are visible in the diagrams, which depend on the discretization of the FEM-mesh, and may be a consequence of the Lagrange method. Another difference is a small asymmetry in the traction, which results from the shift of the stick area. This is clearly visible for the frictional traction marked with crosses at $a^*=5.4$, where the tip of the traction is larger on the left side. The reason of this discrepancy is the analytical assumption of an undeformed contact surface, which neglects the geometrical non-linearity produced by the moving stick zone.

The interior stress field of the numerical (markers) and analytical solution (full line) is illustrated in Figs. 6 and 7 for the normal loading of step 3 ($a=6.2$). Figs. 8
and 9 compare results of step 5 in the regime of partial slip \((a^*=5.4)\). The maximum and the form of the contours are the same for both methods, but the discrete FEM mesh produces some corners in the contours.

Finally, the numerical (markers) and analytical (full line) surface displacements are compared in Figs. 10 & 11. The numerical value for the normal displacement of the flat contact region was superposed on the analytical solution in Fig. 10. This value can alternatively be calculated from the Muskhelishvili potential, when the strain is expressed as a function of the stress \((\varepsilon_z=\sigma_z/E)\). Integration of the strain \(\varepsilon_z\) along the \(z\)-axis gives the displacement \(u_z\), with an initial value of zero at \(x=0, z=R_G=50\) mm, where the half-plane is fixed. The tangential displacement in Fig. 10 was fixed at the end of the \(x\)-axis \(x=R_G=50\) mm, which requires no further calculation. It can be seen that the displacements agree very well.

It may be concluded that for small displacements the FEM model agrees very well with the theory, as long as the geometrical and material nonlinearities can be neglected.

References